(or gain G) required by H_∞ analysis for robust stability of the uncertain closed-loop system with $\|\Delta\|_\infty \le \delta$. This follows from Eq. (8) and is given by

$$\zeta = \frac{G}{2\sqrt{MK}} = \frac{1}{\sqrt{2}}\left[1 - \sqrt{1-(\delta/K)^2}\right]^{1/2} \tag{9}$$

It can be seen from Fig. 5 that as the uncertainty level δ/K increases, H_∞ analysis requires a substantial increase in the nominal closed-loop damping ratio (and thus gain G) to guarantee robust stability.

To illustrate the ramifications of this result, we can determine the minimal damping ratio ζ of the nominal closed-loop system *actually* required to guarantee a worst-case closed-loop damping ratio δ_0 with Δ assumed to be a constant real perturbation satisfying $|\Delta| \le \delta$. To do this, rewrite Eq. (1) as

$$\ddot{x}(t) + 2\zeta_a\omega_{na}\dot{x}(t) + \omega_{na}^2 x(t) = 0 \tag{10}$$

where the actual closed-loop damping ratio ζ_a and natural frequency ω_{na} satisfy

$$2\zeta_a\omega_{na} \triangleq \frac{G}{M}, \qquad \omega_{na}^2 \triangleq \frac{K+\Delta}{M} \tag{11}$$

which imply that the actual closed-loop system damping ratio ζ_a is given by

$$\zeta_a = \frac{G}{2\sqrt{M(K+\Delta)}} \tag{12}$$

Requiring $\zeta_a \ge \zeta_0$ for all $|\Delta| \le \delta$, it follows from Eq. (12) with $\Delta = \delta$ (worst-case damping) that

$$\zeta = \frac{G}{2\sqrt{MK}} \ge \zeta_0\sqrt{1+(\delta/K)} \tag{13}$$

The lower bound in Eq. (13) is plotted in Fig. 5 for two typical cases of ζ_0 corresponding to 5 and 10%. Comparing these curves with the small gain result clearly shows the conservatism of H_∞ theory for constant real parameter uncertainty.

Positive Real Parameter Uncertainty Model

In this section we consider an alternative uncertainty model for the uncertainty Δ. For this model we shift the nominal value of K so that K is positive but arbitrarily close to zero and let the real constant perturbation of K be denoted by Δ where now $0 \le \Delta \le \infty$. Again Fig. 2 can be used to represent this situation. Now, however, we replace Δ by Δ/s and $\tilde{P}$ by $s\tilde{P}$ in Fig. 2. Next note that the effective uncertainty block Δ/s is positive real since $\Delta/j\omega + \Delta/(-j\omega) = 0$, $\omega \in \mathbb{R}$, and, furthermore, the effective plant $s\tilde{P}$ is strictly positive real since

$$j\omega\tilde{P}(j\omega) + [j\omega\tilde{P}(j\omega)]^* = \frac{4\zeta\omega_n\omega}{(\omega_n^2-\omega^2)+(2\zeta\omega_n\omega)^2} > 0, \qquad \omega \in \mathbb{R}$$

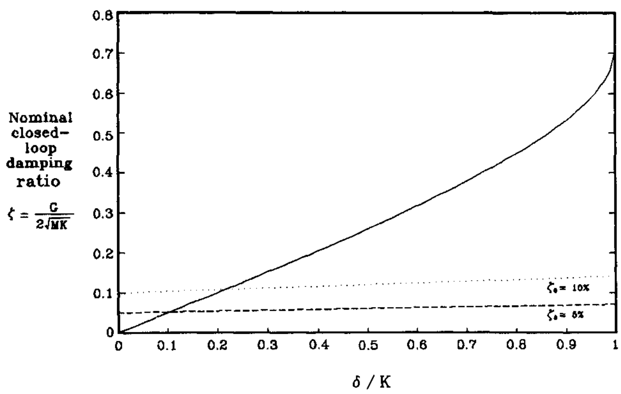

Fig. 5 Nominal closed-loop damping vs stiffness uncertainty.

Hence by the positivity theorem[2] the system is robustly stable for all $\Delta \in [0, \infty)$. Consequently, the closed-loop system is guaranteed to be unconditionally stable for all constant positive velocity feedback gains G. Thus the positive real uncertainty model is, for this example, nonconservative with respect to constant real parameter uncertainty.

Conclusion

We have shown by means of a lightly damped oscillator example with uncertain stiffness that small gain modeling of constant real parameter uncertainty can be extremely conservative. An alternative uncertainty modeling approach involving positive real transfer functions and the positivity theorem was shown to be significantly less conservative.

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Model Order Effects on the Transmission Zeros of Flexible Space Structures

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Introduction

THE dynamics of flexible space structures (FSS) are generally quite poorly known before launch. In particular, most preflight dynamic analysis and controller design for these vehicles is based on approximate finite-dimensional models obtained by finite element methods. However, flexible structures are distributed parameter systems, and so are essentially infinite dimensional. Two implications of this are that the choice of dimension for an approximate finite element model of an FSS is somewhat arbitrary, and that only the lower-frequency approximate modes will tend to be accurate estimates for the corresponding true values.

Recently, it has been shown[1] that the transmission zeros[2] of any finite-dimensional model for a flexible structure with

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compatible (physically collocated and coaxial) sensors and actuators are closely related to the poles, i.e., the natural frequencies and damping ratios, of the structure. Furthermore, the sensitivities of the zeros of such a model to perturbations in its parameters, the dimension of the model being kept fixed, have similarly been proved[3] to be closely related to the sensitivities of the poles. This result holds important practical implications for the pole/zero cancellation vibration suppression technique studied in Ref. 4; one consequence is that the poles of the closed-loop system can be made approximately as robust as were the open-loop poles. However, the question of the sensitivity of the zeros to changes in model dimension has not yet been studied.

This Note investigates this question and proves that a result entirely analogous to the classical Rayleigh-Ritz[5] convergence property for the poles of a flexible structure also applies for its zeros, so long as compatible sensor/actuator pairs are used. Thus, the zeros of a finite-dimensional model of such a structure always bound the true zeros from above, and converge monotonically to these values as model order is increased. Furthermore, the low-frequency fundamental zeros, i.e., those of the greatest physical significance, are the first to converge. One consequence of this result is that a pole/zero cancellation controller designed for a truncated model of a given FSS will cancel the fundamental zeros of the true system quite accurately, doing somewhat less well with the higher-frequency zeros, which are not modeled so precisely. This is equivalent to saying that the low-frequency modes of the closed-loop system, i.e., those that tend to be the most readily excited by disturbances and maneuvers, will appear only weakly in the measured system output. Thus, model order uncertainty does not significantly reduce the performance obtained by a pole/zero cancellation control scheme. A simple cantilever beam system is used to illustrate the new zeros convergence result, as well as the more limited relation that holds in the noncollocated case.

Flexible Space Structure Finite-Dimensional Models

Consider an approximate model of dimension n for the dynamics of a nongyroscopic, noncirculatory FSS with m compatible sensor/actuator pairs. [By compatible, we mean that the direction of the linear (angular) motion measured by each sensor is the same as that of the force (torque) applied by the actuator that is collocated with it.] This model can be written as

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= Vu \\ y &= D_r V^T \dot{q} + D_d V^T q \end{aligned} \tag{1}$$

where q is the vector of generalized coordinates, u that of applied actuator inputs, and y that of sensor outputs. The mass, stiffness, and damping matrices of the structure satisfy $M = M^T > 0$, $K = K^T \geq 0$, and $C = C^T \geq 0$, respectively; the control influence matrix V is of full column rank; and the square matrices D_r and D_d are such that (D_r, D_d) is of full row rank.

Taking the Laplace transform of Eqs. (1) yields the polynomial matrix representation[6]

$$\begin{aligned} P(s)q(s) &= Vu(s) \\ y(s) &= D(s)V^T q(s) \end{aligned} \tag{2}$$

for the given FSS model, where $P(s) = s^2M + sC + K$ and $D(s) = sD_r + D_d$. Note that the matrix $P(s)$ is symmetric. Thus, Eqs. (2) respect the second-order symmetric nature of flexible structures; a state-space model for Eqs. (1) would not.

The transfer matrix of this model is the rational matrix $T(s)$ satisfying $y(s) = T(s)u(s)$. Equations (2) clearly imply that $T(s) = D(s)V^T P(s)^{-1} V$, and so the poles (or eigenvalues) of this model are those s_i for which $P(s_i)$ is singular. Similarly, the transmission zeros[2] are those s_i that make $T(s)$, or equivalently the system matrix

$$S(s) = \begin{pmatrix} P(s) & V \\ -D(s)V^T & 0 \end{pmatrix} \tag{3}$$

singular. [Strictly speaking, $S(s)$ defines the invariant zeros, but these are precisely the transmission zeros for any fully controllable and observable system; this is assumed to be the case here.] Transmission zeros are essentially the eigenvalues corresponding to constrained vibration modes, with the constraint being simply that these modes must produce identically zero measured outputs at all sensors. The fact that the system matrix condition is equivalent to this can be seen by noting that $S(s)(-q^T(s), u^T(s))^T = (y^T(s), 0)^T$. Thus, an identically zero $y(s_i)$ can be obtained for nonzero $q(s_i)$ and $y(s_i)$ if and only if $S(s_i)$ is singular.

The transmission zeros of flexible structure models exhibit far more interesting properties than do the zeros of general linear systems. If damping is assumed to be modal, as is usually the case, then[1] the poles of the FSS define a portion of the left half-plane in which all zeros must lie. This result, which holds for any distribution of compatible sensor/actuator pairs, consists of upper and/or lower bounds on the real and imaginary parts, moduli, and damping ratios of the zeros in terms of the corresponding pole quantities. It is probably best described graphically (see Fig. 1 of Ref. 1) and can be regarded as a generalization of the well-known fact[7,8] that the zeros of a single input/single output undamped structure alternate with its poles up the imaginary axis.

Convergence of Flexible Space Structure Zeros

We now examine how these zeros are affected as the dimension of the model Eqs. (1) is increased from n to $n + 1$. As a first step, Eq. (3) must be transformed to the canonical form used in Ref. 9 for the computation of FSS zeros. This is based on the QR decomposition[10] of V: if we write $V = QR = Q_1R_1$ with $R = (R_1^T, 0)^T$ upper triangular and $Q = (Q_1, Q_2)$ orthogonal, then

$$\hat{S}_n(s) = \operatorname{diag}\{Q, I\}^T S(s) \operatorname{diag}\{Q, I\} = \begin{pmatrix} Q^TP(s)Q & Q^TV \\ -D(s)V^TQ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} Q_1^TP(s)Q_1 & Q_1^TP(s)Q_2 & R_1 \\ Q_2^TP(s)Q_1 & Q_2^TP(s)Q_2 & 0 \\ -D(s)R_1^T & 0 & 0 \end{pmatrix} \tag{4}$$

(The subscript n has been added to denote the model dimension explicitly.) The block triangular structure of $S_n(s)$ then implies that the transmission zeros of Eqs. (1) are just those s_i that make $D(s)$ singular (the sensor zeros), together with the $(n - m)$ conjugate pairs that make $Q_2^TP(s)Q_2$ singular (the structural zeros).

Increasing model dimension by 1 amounts to adding a final row and column to $P(s)$ and row to V, and so the resulting system matrix can be written as

$$S_{n+1}(s) = \begin{pmatrix} Q^TP(s)Q & p(s) & R \\ p^T(s) & \pi(s) & v^T \\ -D(s)R & -D(s)v & 0 \end{pmatrix} \tag{5}$$

This matrix does not exhibit the zeros of this model in any clear way because of the presence of the nonzero vectors v^T and $-D(s)v$. Fortunately, though, orthogonal row and column operations based on Givens rotations[10] can be used to make these vectors zero while preserving the upper triangular form of R and symmetry of $Q^TP(s)Q$. A Givens rotation between rows 1 and $(n + 1)$ of $S_{n+1}(s)$ can always be found that will make zero the first entry of v^T; note that this will also alter the remainder of these two rows. As R_1 is upper triangular, any rotation between row 2 and the new row $(n + 1)$ will keep the first entry of the transformed v^T zero. Such a rotation can always be found that will also make zero the second entry of v^T. Proceeding in this manner, the entire vector can clearly be zeroed, while the same Givens rotations applied between columns $1, \ldots, m$ and $(n + 1)$ will similarly reduce $-D(s)v$ to zero. The resulting transformed system matrix is

$$\hat{S}_{n+1}(s) = \begin{bmatrix} \underline{Q_1^T P(s) Q_1} & \underline{Q_1^T P(s) Q_2} & \underline{p_1(s)} & \underline{R_1} \\ Q_2^T P(s) Q_1 & Q_2^T P(s) Q_2 & \underline{p_2(s)} & 0 \\ \underline{p_1^T(s)} & \underline{p_2^T(s)} & \underline{\pi(s)} & 0 \\ -D(s)\underline{R_1^T} & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

where an underline denotes a quantity that differs from the corresponding one in $S_{n+1}(s)$.

Inspection of the matrix $\hat{S}_{n+1}(s)$ clearly shows that the sensor zeros are unaffected by the increase in model order. Furthermore, the second-order symmetric submatrix that defines the structural zeros just consists of $Q_2^T P(s) Q_2$ (which defined the structural zeros of the model of dimension n) together with an additional row and column. But this is precisely the type of relationship that holds between the denominator matrices of structural models of successive dimension and that leads to the attractive Rayleigh-Ritz[5] convergence properties of the poles of such models. These results state that the low-frequency poles, i.e., those of the greatest physical interest, are the first to converge as n is increased. Furthermore, the magnitudes of the finite-dimensional poles bound the true poles from above and converge monotonically to them as model order increases. We have therefore shown that the zeros of any flexible structure with compatible sensors and actuators exhibit these same attractive convergence properties, a very satisfactory result.

The fact that the preceding derivation is restricted to compatible sensors and actuators is not actually a shortcoming of the analysis. Rather, it simply reflects the very limited properties that exist concerning the zeros of noncollocated structures. For instance,[11] the number of structural zeros is then no longer simply $2(n - m)$; furthermore, right half-plane zeros may occur, making the control problem considerably more challenging than in the compatible case.

However, one general conclusion can still be drawn concerning the zeros of a noncollocated structure as n is increased: the low-frequency zeros will again converge fastest. To see that this is the case, consider the following argument. The zeros of an FSS model are the eigenvalues corresponding to forced vibration modes that produce identically zero sensor outputs. If a new high-frequency natural mode is added to the model, this mode will clearly not be excited to any great extent by the forcing corresponding to a low-frequency zero. The model of order $(n + 1)$ will therefore exhibit only a small measured output when excited at the frequency of a fundamental zero of the model of order n. A small shift in this forcing frequency is all that will be required to produce an identically zero output; in other words, the fundamental zeros of the two models will not differ greatly. However, it should be noted that no guarantee of monotonic convergence can now be given. Indeed, the noncollocated example of the following section demonstrates nonmonotonic convergence.

Example

The zeros convergence results just proved will now be illustrated by application to a simple flexible structure. Consider vertical vibration of a uniform undamped cantilever beam of length 25 m, width 0.1 m, and depth 0.01 m, constructed of aluminum (density 2.7×10^3 kg/m^3, Young's modulus 7.0×10^{10} N/m^2), with a single linear displacement sensor and force actuator collocated at the free tip. If this beam is represented by truncated true modal models [a special case of Eqs. (1) with $M = I$, $C = 0$, and $K = \mathrm{diag}(\omega_i^2)$] of various dimensions n, then the resulting zeros are purely imaginary and have magnitudes (in radians per second) as given in Table 1. Those figures of each zero estimate that differ from the corresponding preceding estimate are underlined for clarity. Also given, as the final row of the table, are the true zeros obtained from the classical continuous (partial differential equation) solution. It can be seen that the finite-dimensional zero estimates are monotonic decreasing for increasing n, as expected. Furthermore, the lowest-frequency zero does indeed converge fastest, followed by the second-lowest, etc., just as predicted by the new result.

Finally, consider the same beam and actuator arrangement, but with the sensor shifted to 7.5 m from the free tip. The zeros of this undamped structure still occur in positive/negative pairs, as in the compatible sensor/actuator case. However, one of these pairs is now purely real, giving a right half-plane zero characteristic of noncollocated structures. The magnitudes of the zeros obtained for various model orders are given in Table 2, together with a notation indicating whether the zero pair is real or imaginary. It can be seen that the fundamental zeros converge fastest, as argued in the last section. Also, the values of $|z_1|$ to $|z_3|$ obtained for $n = 4$ and 5 demonstrate the nonmonotonic convergence that was expected for such structures.

Table 1 Transmission zero magnitudes[a]

n	$\lvert z_1 \rvert$	$\lvert z_2 \rvert$	$\lvert z_3 \rvert$	$\lvert z_4 \rvert$
2	0.371061			
3	0.365171	1.205736		
4	0.363704	1.187535	2.511281	
5	0.363172	1.181428	2.480151	4.285762
∞	0.362602	1.175063	2.451675	4.192510

[a]The figures of each zero estimate that differ from the corresponding preceding estimate are underlined.

Table 2 Noncollocated zero magnitudes[a]

n	$\lvert z_1 \rvert$, imaginary	$\lvert z_2 \rvert$, real	$\lvert z_3 \rvert$, imaginary	$\lvert z_4 \rvert$, imaginary
2	0.756002			
3	0.672662	1.375281		
4	0.667099	1.185020	2.331265	
5	0.668173	1.219393	2.358087	5.347335
∞	0.669944	1.290077	2.393699	5.015097

[a]The figures of each zero estimate that differ from the corresponding preceding estimate are underlined.

Conclusions

This Note has shown that the zeros of finite-dimensional models for a flexible structure with compatible sensors and actuators exhibit precisely the same Rayleigh-Ritz convergence properties as do the poles of such models. In particular, the model zeros always converge monotonically from above to the true values, with the low-frequency zeros being the first to converge. If noncollocated sensors and actuators are used, the fundamental zeros still converge fastest, but monotonicity is no longer guaranteed. These results were illustrated by application to a cantilever beam example.

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